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# Two kinds of iterative solutions for generalized sombrero-shaped potential in $N$ -dimensional space

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## Abstract

Based on two different iteration procedures the groundstate wavefunctions and energies for  $N$ -dimensional generalized Sombrero-shaped potentials are solved. Two kinds of trial functions for the iteration procedure are defined. The iterative solutions converge nicely on consistent results for different choices of iteration procedures and trial functions.

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## 1. Introduction

It is well known that the problem of a non-relativistic particle moving in an  $N$ -dimensional Sombrero-shaped potential provides a prototype sample of the spontaneous symmetry breaking mechanism. This problem is solved properly based on an iterative method developed by Friedberg *et al* [1]. However, the usual Sombrero-shaped potential provides no information about the symmetry restoration. Recently, the generalized radially symmetric Sombrero-shaped potential in the  $N$ -dimensional space has been proposed by Jackiw [2]:

$$V(r) = \frac{1}{2}g^2(r^2 - r_0^2)^2(r^2 + Ar_0^2), \quad (1)$$

where  $r_0$  is related to the dimension  $N$  by  $r_0^4 = (2 + N)/3$ ,  $g^2$  and  $A$  are arbitrary constants. This potential involves the sixth power of the variable and has a more complicated structure, and also gives richer behavior of the shape of wavefunctions. This will be shown in this paper in more detail.

A special simple form of this kind of potentials with zero eigenvalue for its groundstate has been applied by G 't Hooft *et al* [3] for the discussion about the possibility of the existence of a new kind of transformation from real to imaginary spacetime variables. The invariance of this transformation needs to have a zero eigenvalue for the groundstate. A more general discussion about the potential giving a zero eigenvalue groundstate is given in [4].

Jackiw also challenged the author of [1] to apply the iterative method for solving the Schrödinger equation with a generalized Sombbrero-shaped potential. Specially for a simple case when the analytical solution of the groundstate exists, choosing a trial function with maxima at a finite  $r$  the iteration should reach the exact solution with only one maximum at  $r = 0$ . His question is properly answered by the three authors for the one-dimensional case [5]. The same problem will be solved for the  $N$ -dimensional generalized Sombbrero-shaped potential in this paper. The corresponding Schrödinger equation for the groundstate radial wavefunction is

$$\left(-\frac{1}{2r^{2k}} \frac{d}{dr} r^{2k} \frac{d}{dr} + V(r)\right) \psi(r) = E \psi(r) \tag{2}$$

with  $k = (N - 1)/2$ . The boundary conditions are

$$\psi(\infty) = 0 \quad \psi'(0) = 0. \tag{3}$$

When  $g = 1$  and  $A = 2$  the solution of the groundstate has an analytical form as  $\psi(r) = e^{-r^4/4}$  with the eigenvalue  $E_0 = r_0^6$ . However, for arbitrary  $g$  and  $A$  the groundstate wavefunction has no analytical form. In the following two iterative solutions for the groundstate of (2) are presented. To apply the iterative methods we introduce the trial function  $\phi(r)$  satisfying another Schrödinger equation

$$\left(-\frac{1}{2r^{2k}} \frac{d}{dr} r^{2k} \frac{d}{dr} + V(r) - h(r)\right) \phi(r) = gE_0\phi(r) = (E - \Delta)\phi(r), \tag{4}$$

where  $h(r)$  and  $\Delta$  are the corrections of the potential and groundstate energy. Starting from this trial function  $\phi(r)$  we perform two iterative procedures. Define the exact wavefunction as

$$\psi(r) = f(r)\phi(r) = e^{-\tau(r)}\phi(r). \tag{5}$$

The iteration performed for  $f(r)$  and  $\Delta$  is named  $f$ -iteration [1] and that performed for  $\tau(r)$  and  $\Delta$  is named  $\tau$ -iteration [6] in this paper. For the  $f$ -iteration two iterative series of  $\{f_n(r)\}$  and  $\{\Delta_n^f\}$ ,  $n = 0, 1, \dots$ , are introduced with  $f_0(r) = 1$  and  $\Delta_0^f = 0$ . For the  $\tau$ -iteration two iterative series of  $\{\tau_n(r)\}$  and  $\{\Delta_n^\tau\}$ ,  $n = 0, 1, \dots$ , are defined with  $\tau_0(r) = 0$  and  $\Delta_0^\tau = 0$ . The iterations for these two sets can be performed according to the following equations. For the  $f$ -iteration we have [1]

$$\Delta_n^f = \frac{\int_0^\infty r^{2k} \phi^2(r) h(r) f_{n-1}(r) dr}{\int_0^\infty r^{2k} \phi^2(r) f_{n-1}(r) dr}, \tag{6a}$$

$$f_n(r) = 1 - 2 \int_{r_c}^r \frac{dy}{y^{2k} \phi^2(y)} \int_{r_c}^y x^{2k} \phi^2(x) (\Delta_n^f - h(x)) f_{n-1}(x) dx, \tag{6b}$$

where  $r_c$  could be chosen as  $r_c = 0$  or  $r_c = \infty$  and the normalization is chosen as  $f_n(r_c) = 1$ . As for the  $\tau$ -iteration we have [6]

$$\Delta_n^\tau = \frac{\int_0^\infty r^{2k} \phi^2(r) (h(r) - \frac{1}{2}(\tau'_{n-1}(r))^2) dr}{\int_0^\infty r^{2k} \phi^2(r) dr}, \tag{6c}$$

$$\tau'_n(r) = 2r^{-2k} \phi^{-2}(r) \int_0^r y^{2k} \phi^2(y) [(\Delta_n^\tau - h(y)) + \frac{1}{2}(\tau'_{n-1}(y))^2] dy \tag{6d}$$

where  $\tau'(r) = \frac{d\tau}{dr}$ . The detailed derivation is given in the appendix. To ensure the convergency of the iterative methods it is necessary to construct the trial function in such a way that the perturbed potential  $h(r)$  is always positive (or negative) and finite everywhere. Specially,  $h(r) \rightarrow 0$  when  $r \rightarrow \infty$ . In the following we construct two different trial functions for the iteration procedures.

## 2. Trial functions

### 2.1. Trial function I

Introduce

$$\phi(r) = e^{-S_0(r)}. \tag{7}$$

Now substituting (7) into (4) we obtain the equation for  $S_0(r)$ :

$$S_0'(r)^2 - \frac{2k}{r}S_0'(r) - S_0''(r) = 2(V(r) - h(r) - gE_0). \tag{8}$$

Therefore

$$h(r) + gE_0 = V(r) - \frac{1}{2}(S_0'(r)^2 - \frac{2k}{r}S_0'(r) - S_0''(r)). \tag{9}$$

For a finite  $h(r)$  it should not include terms with positive power of  $r$ . Since the highest order of  $r$ -power in the potential is 6 and  $V(r)$  has only even powers of  $r$ , we first assume

$$S_0(r) = (ar^4 + cr^2 + e) + m \log(\alpha r^2 + 1). \tag{10}$$

Substituting (10) into (8), to cancel the  $r^6$  term we have  $a = g/4$ . Coefficients  $e$  and  $\alpha$  only change the normalization and we simply set  $e = 0, \alpha = 1$  and finally obtain

$$S_0(r) = \left(\frac{g}{4}r^4 + cr^2\right) + m \log(r^2 + 1). \tag{11}$$

To cancel the terms with  $r^4$  and  $r^2$  we set

$$c = \frac{1}{4}g(A - 2)r_0^2 \quad \text{and} \quad m = \frac{1}{4}(g + 3)r_0^4 - \frac{1}{16}g(A + 2)^2r_0^4 \tag{12}$$

and obtain

$$h(r) = 2m(m + 1)\frac{1}{(r^2 + 1)^2} + (mg(A - 2)r_0^2 - 2mg + 2mk - 2m^2 - m)\frac{1}{r^2 + 1} \tag{13}$$

$$gE_0 = \frac{1}{2}Ag^2r_0^6 + 2mg + (2k + 1 - 4m)\frac{1}{4}g(A - 2)r_0^2. \tag{14}$$

When  $g = 1$  and  $A = 2$  we have  $m = 0, h(r) = 0$  and the trial function is just the exact solution of the Schrödinger equation. To look in more detail the behavior of the trial function we choose  $g = 1$  and see the change of the trial function with the parameter  $A$ . When  $A = 2$  our trial function is just the exact solution with the maximum of the wavefunction at  $r = 0$ . For  $A \neq 2$  we always have  $h(r) < 0$  and finite, and  $h(r) \rightarrow 0$  when  $r \rightarrow \infty$ . This ensures the convergency of the iterative procedure. When  $A < 2$  the potential is more centered at  $r = 0$  and the trial function keeps its maximum at  $r = 0$ . When  $A > 2$  the potential is more like a double-well and the trial function has maxima at  $r \neq 0$ .

### 2.2. Trial function II

We can introduce another trial function

$$\phi(r) = \left(\frac{r_0 + a}{r + a}\right)^k e^{-gS_0(r) - S_1(r)}, \quad k = (N - 1)/2 \tag{15}$$

satisfying the Schrödinger equation (4) and the boundary condition

$$\phi(\infty) = 0.$$

The parameter  $a$  in (15) is fixed to ensure the boundary condition

$$\phi'(0) = 0,$$

namely

$$gS'_0(0) + S'_1(0) + \frac{k}{a} = 0. \tag{16}$$

Substituting (15) into (4), we compare terms with same power of  $g$ . From  $g^2$ -terms we obtain

$$S'_0(r) = \sqrt{2v} = (r^2 - r_0^2)\sqrt{r^2 + Ar_0^2}. \tag{17}$$

To ensure the  $h(r)$  satisfying the convergence condition,  $S_1(r)$  is defined in a special way to prevent terms with positive powers of  $r$  from presenting in  $h(r)$ . For  $g^1$  terms we have

$$-\frac{1}{2}(2S'_0S'_1 - S''_0 - 2ka\sqrt{r^2 + Ar_0^2})|_{r=r_0} - ka^2 = E_0. \tag{18}$$

Introducing

$$E_0 = E_0^{(1)} + E_0^{(2)} + E_0^{(3)} \tag{19}$$

and defining

$$E_0^{(3)} = -ka^2 \tag{20}$$

we write

$$S'_0S'_1 = (\frac{1}{2}S''_0 - E_0^{(1)}) + (ka\sqrt{r^2 + Ar_0^2} - E_0^{(2)}).$$

Since  $S'_0(r_0) = 0$  we obtain

$$E_0^{(1)} = \frac{1}{2}S''_0(r_0) = r_0^2\sqrt{1 + A}, \tag{21}$$

$$E_0^{(2)} = kar_0\sqrt{1 + A} \tag{22}$$

and

$$S'_1 = (\frac{1}{2}S''_0 - E_0^{(1)})/S'_0 + (ka\sqrt{r^2 + Ar_0^2} - E_0^{(2)})/S'_0. \tag{23}$$

Substituting  $S'_0(r)$  into (23) we have explicitly

$$S'_1(r) = \frac{r^2 + (1 + A)r_0^2}{\sqrt{r^2 + Ar_0^2}(r\sqrt{r^2 + Ar_0^2} + r_0^2\sqrt{1 + A})} + \frac{r}{2(r^2 + Ar_0^2)} + \frac{ka}{\sqrt{r^2 + Ar_0^2}(\sqrt{r^2 + Ar_0^2} + r_0\sqrt{1 + A})}. \tag{24}$$

The expression for  $h(r)$  is

$$h(r) = \frac{1}{2}(S_1^2 - S_1'') + \frac{1}{2} \frac{k(k+1)}{(r+a)^2} - \frac{ka}{r(r+a)}S'_1 - \frac{k^2}{r(r+a)} + kag(r_0^2 - a^2) \frac{\sqrt{r^2 + Ar_0^2}}{r(r+a)} + ka^2g \frac{Ar_0^2}{r(\sqrt{r^2 + Ar_0^2} + r)}. \tag{25}$$

Substituting  $S'_0(0)$  and  $S'_1(0)$  into (16) we obtain an equation for the parameter  $a$

$$ka^2 + (r_0\sqrt{1 + A} - gr_0^5A)(\sqrt{A} + \sqrt{1 + A})a + kr_0^2(A + \sqrt{A(1 + A)}) = 0. \tag{26}$$

For the above equation to have real solutions of  $a$  the following restriction is put on the parameters  $g$  and  $A$ :

$$(\sqrt{1+A} - gAr_0^4)^2 r_0^2 (\sqrt{A} + \sqrt{1+A}) - 4\sqrt{A}k^2 r_0^2 \geq 0. \tag{27}$$

For example, for  $N = 3$ , i.e.  $k = 1$ , (27) requires  $g > 0.922$  when  $A = 2$  and  $A > 1.81$  when  $g = 1$ . When (27) cannot be fulfilled the condition  $\phi'(0) = 0$  can be satisfied by introducing the trial function as

$$\phi_{\text{rev}}(r) = \phi(r) + \xi\phi_-(r) \quad \text{for } r < r_0 \tag{28a}$$

and

$$\phi_{\text{rev}}(r) = (1 + \xi\phi_-(r_0)/\phi(r_0))\phi(r) \quad \text{for } r > r_0 \tag{28b}$$

where  $\phi_-(r)$  is defined as

$$\phi_-(r) = \left(\frac{r_0 + a}{r + a}\right)^k e^{-gS_0(-r) - S_1(r)}. \tag{29}$$

The parameter  $\xi$  is fixed to satisfy the condition  $\phi'_{\text{rev}}(0) = 0$ , namely

$$\phi'(0) + \xi\phi'_-(0) = 0. \tag{30}$$

Correspondingly the Schrödinger equation satisfied by  $\phi_{\text{rev}}(r)$  is

$$\left(-\frac{1}{2r^{2k}} \frac{d}{dr} r^{2k} \frac{d}{dr} + V(r) - h_{\text{rev}}(r)\right) \phi_{\text{rev}}(r) = gE_0 \phi_{\text{rev}}(r), \tag{31}$$

where  $h_{\text{rev}}(r) = h(r)$  for  $r > r_0$  and

$$h_{\text{rev}}(r) = h(r) + 2g\xi \left( E_0 + ka \frac{a^2 - r_0^2}{r(r+a)} \sqrt{r^2 + Ar_0^2} - ka^2 \frac{Ar_0^2}{r(\sqrt{r^2 + Ar_0^2} + r)} \right) \phi_-(r)/\phi_{\text{rev}}(r) \tag{32}$$

for  $r < r_0$ . It is interesting to note that the conditions (26) and (30) for  $\phi'(0) = 0$  also ensure  $h(r)$  and  $h_{\text{rev}}(r)$  to be finite when  $r \rightarrow 0$ , which is necessary for the convergency of the iteration procedure.

By integrating (17) and (25) we obtain  $S_0(r)$  and  $S_1(r)$  as

$$S_0(r) = \frac{1}{8}r\sqrt{r^2 + Ar_0^2}(2r^2 + Ar_0^2 - 4r_0^2) - \frac{1}{8}(A^2r_0^4 + 4Ar_0^4) \ln(r + \sqrt{r^2 + Ar_0^2}) \tag{33}$$

$$S_1(r) = \ln(r + r_0) + \frac{1}{4} \ln(r^2 + Ar_0^2) + \left(\frac{1}{2} + \frac{ka}{2r_0}\right) \ln \frac{\sqrt{1+A}\sqrt{r^2 + Ar_0^2} + r + Ar_0}{\sqrt{1+A}\sqrt{r^2 + Ar_0^2} - r + Ar_0}. \tag{34}$$

Substituting them into (15) or (28)–(29) gives the final expression of the trial functions. From (25) we can also reach

$$\begin{aligned} \frac{1}{2}(S_1'' - S_1') &= \frac{\gamma}{8(r^2 + Ar_0^2)^2(\alpha + \beta)} + \frac{ka}{2r(r+a)(r^2 + Ar_0^2)} \frac{\gamma'}{\alpha' + \beta'} \\ &+ \frac{k^2 a^2}{2(r^2 + Ar_0^2)(\sqrt{r^2 + Ar_0^2} + r_0\sqrt{1+A})^2} \\ &+ \frac{kar(2\sqrt{r^2 + Ar_0^2} + r_0\sqrt{1+A})}{(r^2 + Ar_0^2)^{3/2}(\sqrt{r^2 + Ar_0^2} + r_0\sqrt{1+A})^2} \end{aligned} \tag{35}$$

with

$$\begin{aligned}\gamma &= 225r^8 + 270(1 + 2A)r^6r_0^2 + 3(188A^2 + 216A - 5)r^4r_0^4 \\ &\quad + 36A(8A^2 + 10A - 1)r^2r_0^6 + 4A^2(4A + 1)^2r_0^8, \\ \gamma' &= 9r^4 + 3(4A + 1)r^2r_0^2 + 4A(1 + A)r_0^4, \\ \alpha &= 15r^6 + (18A - 6)r^4r_0^2 + (8A^2 + 12A + 7)r^2r_0^4 + (8A^2 + 2A)r_0^6, \\ \beta &= 8\sqrt{1 + Ar_0^2}r(3r^2 + (2A - 1)r_0^2)\sqrt{r^2 + Ar_0^2}\end{aligned}$$

and

$$\alpha' = r(3r^2 + (2A - 1)r_0^2), \quad \beta' = 2r_0^2\sqrt{1 + A}\sqrt{r^2 + Ar_0^2}.$$

Substituting (35) into (25) and (32) gives the final expressions of  $h$  and  $h_{\text{rev}}$ . With the above results for  $h$ ,  $\phi$  and  $h_{\text{rev}}$ ,  $\phi_{\text{rev}}$  we are ready to perform the iteration procedure.

### 3. Numerical result

Starting from the above-defined two sets of trial functions  $\phi(r)$  and the related  $h(r)$ , we can perform the iterations based on the  $f$ -iteration of (6a) and (6b) or  $\tau$ -iteration of (6c) and (6d). Our numerical results show that although the two iteration procedures look quite different and the two trial functions are defined in different ways the finally obtained wavefunctions and eigenvalues for the groundstate converge nicely on the same final shapes and values. Now we give some more detailed discussions about our results. Let us take  $N = 3$  as an example.

For  $g = 1$  and  $A = 2$

For the trial function I, when  $g = 1$  and  $A = 2$ , as mentioned before, the trial function gives just the exact solution of the groundstate

$$\phi(x) = e^{-r^4/4}$$

with  $E_0 = r_0^6$ . However, for the trial function II, it is necessary to fix the parameter  $a$  in the trial function (15) first by solving (26). It gives

$$a = 4.4267 \quad \text{or} \quad a = 1.2976$$

Performing the iteration based either on the  $f$ -iteration or on the  $\tau$ -iteration the final convergent result of the wavefunction and the eigenvalue of the groundstate are consistent with the exact solution. The trial function and the final exact wavefunction for the groundstate are plotted in figure 1. It is interesting to observe the transition of the shape of the wavefunction for the trial function with maxima at a finite  $r$  to the final convergent one with only one maximum at  $r = 0$  after the iteration procedure, as the exact groundstate wavefunction should be. This answered the question raised by Jackiw [2] in the  $N$ -dimensional case: Even the trial function proposed has its maxima at  $r > 0$  the iteration procedure would still reach the exact solution of the groundstate wavefunction with its only maximum at  $r = 0$ .

#### 3.1. Comparison of the two iteration procedures

In tables 1 and 2, the eigenvalues of the groundstate obtained from the  $\tau$ - and  $f$ -iterations are listed, respectively, based on the two different trial functions for different parameters  $g$  and  $A$ . Comparing the two iteration procedures, it can be seen that the  $\tau$ -iteration is convergent faster

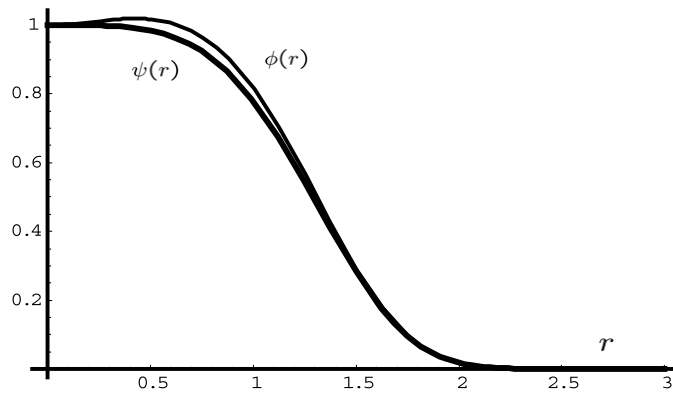


Figure 1. Trial function  $\phi(r)$  and groundstate wavefunction  $\psi(r)$  for  $N = 3$ ,  $g = 1$  and  $A = 2$ .

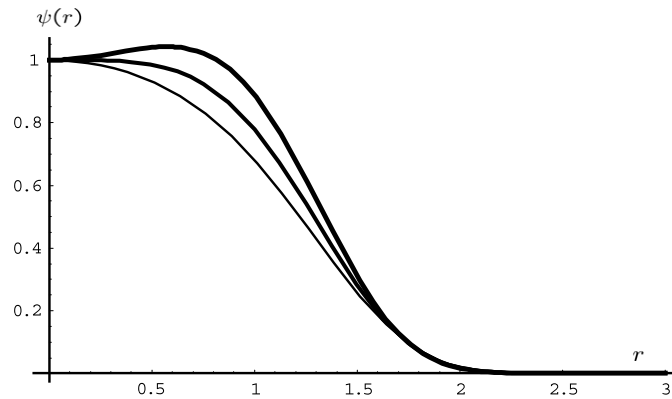
Table 1. Eigenvalues of groundstates for  $N = 3$  based on  $\tau$ -iteration.

$g$	$A$	Trial function	$E_0$	$E_1$	$E_2$	$E_3$	$E_4$	$E_5$
0.5	2	I	1.1629	1.3978	1.3763	1.3772	1.3773	
0.5	2	II	-0.4300	1.3963	1.3763	1.3773	1.3773	
0.93	2	I	2.0237	2.0352	2.0351	2.0351	2.0351	
0.93	2	II	2.0921	2.0457	2.0355	2.0351	2.0351	
1	2	I	2.1517					
1	2	II	-8.6479	2.1523	2.1517	2.1517	2.1517	
2	2	I	3.6066	4.1140	4.1093	4.1094	4.1094	
2	2	II	5.5581	4.1362	4.1123	4.1097	4.1094	
1	1	I	2.5073	1.8400	1.8392	1.8392	1.8392	
1	1	II	-2.3537	1.8920	1.8330	1.8394	1.8393	1.8392
1	1.9	I	2.1225	2.1215	2.1215	2.1215	2.1215	
1	1.9	II	-3.8095	2.1232	2.1215	2.1215	2.1215	
1	3	I	3.5310	2.4630	2.4426	2.4418	2.4418	
1	3	II	3.6773	2.4675	2.4437	2.4419	2.4418	

Table 2. Eigenvalues of groundstates for  $N = 3$  based on the  $f$ -iteration.

$g$	$A$	Trial function	$E_0$	$E_1$	$E_2$	$E_3$	$E_4$	$E_5$
0.5	2	I	1.1629	1.3978	1.3705	1.3786	1.3770	1.3773
0.5	2	II	-0.4300	1.3963	1.3795	1.3775	1.3773	1.3773
0.93	2	I	2.0237	2.0352	2.0351	2.0351	2.0351	
0.93	2	II	2.0921	2.0457	2.0337	2.0352	2.0351	2.0351
1	2	I	2.1517					
1	2	II	-8.6479	2.1523	2.1516	2.1517	2.1517	2.1517
2	2	I	3.6066	4.1140	4.1088	4.1094	4.1094	
2	2	II	5.5581	4.1362	4.0976	4.1108	4.1092	4.1094
1	1	I	2.5073	1.8400	1.8392	1.8392	1.8392	
1	1	II	-2.3537	1.8920	1.8473	1.8402	1.8393	1.8392
1	1.9	I	2.1225	2.1215	2.1215	2.1215	2.1215	
1	1.9	II	-3.8095	2.1232	2.1214	2.1215	2.1215	2.1215
1	3	I	3.5310	2.4630	2.4464	2.4425	2.4419	2.4418
1	3	II	3.6773	2.4675	2.4353	2.4425	2.4417	2.4418





**Figure 2.** groundstate wavefunction  $\psi(r)$  for  $N = 3$ ,  $g = 1$  and  $A = 1$  (thin), 2 (middle) and 3 (thick).

than the  $f$ -iteration. The numerical calculation also takes less time to reach the convergent result for the  $\tau$ -iteration. This can be understood by comparing the formulae (6a) and (6b) for the  $f$ -iteration with (6c) and (6d) for  $\tau$ -iteration. First, in the formula for the energy correction, the denominator changes in each order in the  $f$ -iteration while it needs only to calculate once for the whole  $\tau$ -iteration procedure. Besides, one fold less of integration is needed for each order of iteration in the  $\tau$ -iteration procedure since it is related only to  $\tau'_n$ . These two advantages speed up the numerical calculation of the  $\tau$ -iteration very much. It is shown in the appendix that the two iteration procedures are derived from the same Schrödinger equation; however they choose different unknown functions, namely  $f$  or  $\tau$ , to perform the iteration, therefore, are independent of each other. Although they approach the convergent results in different ways with different convergent speeds, it is clearly shown in two tables that the two iteration procedures do give the same convergent results.

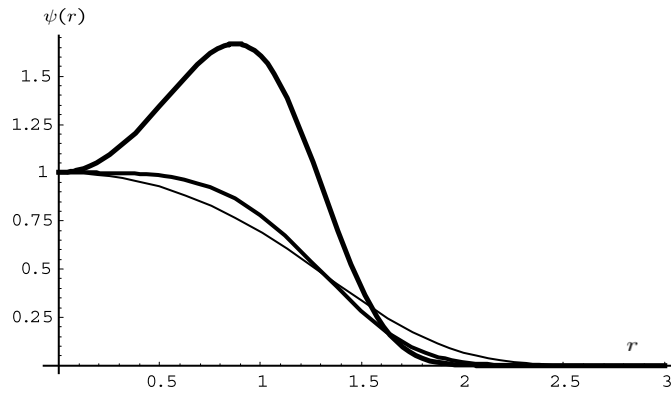
In the following we compare the two trial functions and analyze the change of the wavefunction shapes with parameters of the potential. Since the two iteration procedures give the same convergent results the following discussions are suitable for both procedures.

### 3.2. Comparison of the two trial functions

For the two trial functions, the trial function I is closer to the groundstate and needs less orders of iteration to reach the exact result in most cases. This can clearly be seen for the case of  $g = 1$  and  $A = 2$ . The trial function I has already given the exact solution while for the trial function II which has maxima at  $r > 0$  the exact groundstate wavefunction with its only maximum at  $r = 0$  can be reached only after the iteration. In fact, for different parameters  $g$  and  $A$  the trial functions I always have shapes similar to the exact solution, while the trial functions II differ from the exact ones in their shapes for  $g \leq 1$  and  $A \leq 2$ . Although the iteration process for the two trial functions is quite different, the two iteration procedures with two sets of trial functions always reach the same final results of eigenvalues and groundstate wavefunctions.

### 3.3. Change of the wavefunction shapes with parameters

As examples the obtained groundstate wavefunctions after the iteration procedure are plotted in figures 2 and 3 for  $A = 2$  and  $g = 0.5, 1$  or  $2$ , and for  $g = 1$  and  $A = 1, 2$  or  $3$ , respectively.



**Figure 3.** Groundstate wavefunction  $\psi(r)$  for  $N = 3$ ,  $A = 2$  and  $g = 0.5$  (thin), 1 (middle) and 2 (thick).

It is interesting to see the transition of the form of the obtained groundstate wavefunction from the shape with maximum at  $r = 0$  to that with maxima at a finite  $r$ , becoming a degenerate groundstate, when  $g$  increases from  $<1$ , passing 1 to  $>1$  for  $A = 2$ , or when  $A$  increases from  $<2$ , passing 2 to  $>2$  for  $g = 1$ . Therefore, the groundstate wavefunctions in the region  $g \leq 1$  and  $A \leq 2$  have the shape with only one maximum at  $r = 0$ , while in the region outside the wavefunctions have maxima at a finite  $r$  and the groundstates become degenerate. Their maxima move to larger  $r$  when the parameters  $g$  and  $A$  increase further.

Our results show that the generalized Sombrero-shaped potential not only gives the degenerate groundstate for the properly chosen parameters  $A > 2$  and  $g > 1$ , but also gives the transition of the shape of the wavefunctions from the degenerate groundstate to those with only one maximum at  $r = 0$  by changing the parameters of the potential to  $A \leq 2$  and  $g \leq 1$ . This provides the possibility of studying the relation between the restoration of spontaneous symmetry breaking and the change of the parameters of the potential. It is worthy to look at the problem in more detail and try to relate the parameters of the potential to possible dynamical mechanisms.

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### Appendix

For a particle with unit mass, moving in an  $N$ -dimensional potential  $V(\mathbf{r})$ , the groundstate wavefunction  $\Psi(\mathbf{r})$  satisfies the following Schroedinger equation:

$$\left(-\frac{1}{2}\nabla^2 + V(\mathbf{r})\right) \Psi(\mathbf{r}) = E\Psi(\mathbf{r}). \tag{A.1}$$

Introduce a potential correction  $h(\mathbf{r})$  and define another wavefunction  $\Phi(\mathbf{r})$  satisfying the Schroedinger equation

$$\left(-\frac{1}{2}\nabla^2 + V(\mathbf{r}) - h(\mathbf{r})\right) \Phi(\mathbf{r}) = (E - \Delta)\Phi(\mathbf{r}), \tag{A.2}$$

where  $\Delta$  is the corresponding energy correction. Multiplying (A.2) on the left by  $\Psi(\mathbf{r})$  and (A.1) by  $\Phi(\mathbf{r})$ , their difference gives

$$-\frac{1}{2}\nabla(\Phi\nabla\Psi - \Psi\nabla\Phi) = -(h - \Delta)\Phi\Psi. \tag{A.3}$$

Let

$$\Psi(\mathbf{r}) = \Phi(\mathbf{r})f(\mathbf{r}) = \Phi(\mathbf{r})e^{-\tau(\mathbf{r})}. \quad (\text{A.4})$$

In [1], an equation is introduced for  $(f(\mathbf{r}), \Delta)$ . Here according to the same procedure similar results are deduced for  $(\tau(\mathbf{r}), \Delta)$ . For the convenience of comparison we list both results together in the following. Substituting (A.4) into (A.3) equations for  $(f, \Delta)$  and  $(\tau, \Delta)$  could be obtained as follow:

$$-\frac{1}{2}\nabla(\Phi^2\nabla f) = (-h + \Delta)\Phi^2 f \quad (\text{A.5})$$

and

$$\frac{1}{2}\nabla(\Phi^2\nabla\tau) = (-h + \Delta + \frac{1}{2}(\nabla\tau)^2)\Phi^2. \quad (\text{A.6})$$

The integration over all space for the left-hand side of the above equations is zero. This leads to the expressions of the energy correction related to  $f$  and  $\tau$ , respectively:

$$\Delta = \frac{\int d\mathbf{r} \Phi^2 h f}{\int d\mathbf{r} \Phi^2 f}, \quad (\text{A.7})$$

and

$$\Delta = \frac{\int d\mathbf{r} \Phi^2 [h - \frac{1}{2}(\nabla\tau)^2]}{\int d\mathbf{r} \Phi^2}. \quad (\text{A.8})$$

Introducing

$$\Delta_1^i, \Delta_2^i, \dots, \Delta_n^i, \dots, \quad i = f \text{ or } \tau \quad (\text{A.9})$$

and

$$f_1, f_2, \dots, f_n, \dots \quad \text{or} \quad \tau_1, \tau_2, \dots, \tau_n, \dots \quad (\text{A.10})$$

the two iterative series are defined as

$$\frac{1}{2}\nabla(\Phi^2\nabla\tau_n) = \left(-h + \Delta_n^\tau + \frac{1}{2}(\nabla\tau_{n-1})^2\right)\Phi^2, \quad (\text{A.11})$$

$$\Delta_n^\tau = \frac{\int d\mathbf{r} \Phi^2 [h - \frac{1}{2}(\nabla\tau_{n-1})^2]}{\int d\mathbf{r} \Phi^2} \quad (\text{A.12})$$

and

$$-\frac{1}{2}\nabla(\Phi^2\nabla f_n) = (-h + \Delta_n^f)\Phi^2 f_{n-1}, \quad (\text{A.13})$$

$$\Delta_n^f = \frac{\int d\mathbf{r} \Phi^2 h f_{n-1}}{\int d\mathbf{r} \Phi^2 f_{n-1}}. \quad (\text{A.14})$$

For later convenience the iteration series for  $\{\tau_n, \Delta_n^\tau\}$  defined in equations (A.11) and (A.12), originally introduced in [6], is named the  $\tau$ -iteration, while that for  $\{f_n, \Delta_n^f\}$  given in (A.13) and (A.14), originally introduced in [1], is named  $f$ -iteration in this paper. By introducing the external electrostatic charge distributions

$$\sigma_n^\tau = (-h + \Delta_n^\tau + \frac{1}{2}(\nabla\tau_{n-1})^2)\Phi^2 \quad (\text{A.15})$$

and

$$\sigma_n^f = (-h + \Delta_n^f)\Phi^2 f_{n-1}, \quad (\text{A.16})$$

correspondingly also defining the displacement electric fields

$$D_n^\tau = \frac{1}{2} \Phi^2 \nabla \tau_n \tag{A.17}$$

and

$$D_n^f = -\frac{1}{2} \Phi^2 \nabla f_n, \tag{A.18}$$

where one can define  $\kappa = \Phi^2$  corresponding to the dielectric constant in the usual electrostatic problem. Now equations (A.11) and (A.13) can be expressed by the Maxwell equations of the electrostatic analog problem for  $\tau$ - and  $f$ -iteration [1, 6]:

$$\nabla \cdot D_n^i = \sigma_n^i, \quad \text{for } i = \tau, f. \tag{A.19}$$

If this electrostatic analog problem (A.19) can be solved numerically, starting from the initial conditions

$$\Delta_0^\tau = 0, \quad \tau_0 = 0$$

or

$$\Delta_0^f = 0, \quad f_0 = 1,$$

the corrections of the groundstate energy and wavefunction can be solved by the iteration procedures (A.19) and (A.12) or (A.14). However, the dielectric constant  $\kappa = \Phi^2$  in this problem is not a constant, but changes with  $\mathbf{r}$ . This makes it quite complicated to solve (A.19). Works along this direction are still in progress.

For the radially symmetric potential  $V(r)$  and potential correction  $h(r)$  the problem for solving the groundstate can be simplified and is related only to the radial variable  $r$ . By separating the angular variables [1] (A.1) and (A.2) can be reduced to equations (2) and (4) for the groundstate radial wavefunctions  $\psi(r)$  and  $\phi(r)$ . Multiplying (4) on the left by  $\psi(r)$  and (2) by  $\phi(r)$ , their difference gives

$$-\frac{1}{2r^{2k}} \frac{d}{dr} \left( \psi r^{2k} \frac{d}{dr} \phi - \phi r^{2k} \frac{d}{dr} \psi \right) = (h - \Delta) \phi \psi. \tag{A.20}$$

Let

$$\psi(r) = \phi(r) f(r) = \phi(r) e^{-\tau(r)}. \tag{A.21}$$

The equations for  $(f(r), \Delta)$  and  $(\tau(r), \Delta)$  are deduced as follows:

$$\frac{d}{dr} \left[ r^{2k} \phi^2 \frac{df}{dr} \right] = 2r^{2k} (h - \Delta) \phi^2 f \tag{A.22}$$

and

$$-\frac{d}{dr} \left[ r^{2k} \phi^2 \frac{d\tau}{dr} \right] = 2r^{2k} \left[ h - \Delta - \frac{1}{2} \left( \frac{d\tau}{dr} \right)^2 \right] \phi^2. \tag{A.23}$$

The integration of the left-hand side of equations (A.22) and (A.23) over  $r = 0$  to  $\infty$  is zero, which gives the expressions of the energy correction

$$\Delta = \frac{\int_0^\infty r^{2k} \phi^2(r) h(r) f(r) dr}{\int_0^\infty r^{2k} \phi^2(r) f(r) dr} \tag{A.24}$$

and

$$\Delta = \frac{\int_0^\infty r^{2k} \phi^2(r) \left( h(r) - \frac{1}{2} (\tau'(r))^2 \right) dr}{\int_0^\infty r^{2k} \phi^2(r) dr}, \tag{A.25}$$

with  $\tau' = \frac{d\tau}{dr}$ . Introducing the two iterative series  $\{f_n(r)\}$ ,  $\{\Delta_n^f\}$  and  $\{\tau_n(r)\}$ ,  $\{\Delta_n^\tau\}$  with  $n = 0, 1, \dots$  it is easy to obtain the iteration equations (6a)–(6d) from (A.22)–(A.25).

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